Represntations of Affine Lie Algebras

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1 Preliminaries

1.1 Kac-Moody and Affine Lie Algebras

Kac-Moody algebras are a class of infinite dimensional Lie algebras that has a similar representation theory to that of semi-simple Lie algebras. Namely, admitting s root space decomposition, an associated (ind) Lie group, a casimir element, Borel-Weil-Bott etc. This can be done using a *generalized Cartan matrix* (i.e. dropping the positive definite requirement).

Affine Kac-Moody algebras are a subclass of KM algebras associated to semi-simple Lie algebras, and are defined as a central extension $\hat{\mathfrak{g}}$ (or its polynomial counterpart $\hat{\mathfrak{g}}^{\mathrm{pol}}$) of the loop algebra $\mathfrak{g}\left((t)\right) \coloneqq \mathfrak{g} \otimes \mathbb{C}\left((t)\right)$ (resp. $\mathfrak{g}[t^{\pm 1}]$), with Lie brecket

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$$

The polynomial Kac-Moody algebras can be defined as an abstract Kac-Moody algebra using a Cartan matrix: Start with a semi-simple Lie algebra $\mathfrak{g}$ with a Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq n}$. Add a zero row and column $\hat{A} = (a_{ij})_{0 \leq i,j \leq n}$ with $a_{0,0} = 2, a_{0,j} = -\alpha_j\left(\check{\theta}\right), a_{j,0} = -\theta\left(\check{\alpha}_j\right), j > 0$. $\hat{\mathfrak{g}}^{\mathrm{pol}}$ is the Lie algebra associated to $\hat{A}$. By completing with respect to the t-adic topology, we can get the entire affine Lie algebra $\hat{\mathfrak{g}}$.

A second definition can be given by explicitly describing the central extension. This allows us to relate representation theory of $\mathfrak{g}$ to that of $\hat{\mathfrak{g}}$ as well as to that of the loop group LG.

In this talk we'll use the seconf point of view.

Definition 1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, and κ an invariant symmetric bilinear form on $\mathfrak{g}$. The affine Lie algebra associated to $(\mathfrak{g}, \kappa)$ is the central extension*

$$0 \to \mathbb{C}\mathbf{1} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}\left((t)\right) \to 0$$

with $\mathbf{1}$ central and

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \kappa\left(X, Y\right) \mathrm{Res}_{t=0}\, f dg \cdot \mathbf{1}$$

Such κ's are classified by $H^2\left(\mathfrak{g}, \mathbb{C}\right)$ and can be shown to be one dimensional. In particular, fixing κ_0 the normalized Killing form with $\kappa_0\left(\check{\alpha}, \check{\alpha}\right) = 2$ for any long root α, any other κ will be a scalar multiple of κ_0. The critical form κ_c is defined as

$$\kappa_c = -\frac{1}{2}\,\mathrm{Kil}$$

Definition 2. *A $\hat{\mathfrak{g}}_\kappa$ representation is called smooth if it is killed by $t^{-N}\mathfrak{g}[[t]]$ for $N \gg 0$. Let $\hat{\mathfrak{g}}_\kappa\text{-mod}^\heartsuit$ be the category of smooth $\hat{\mathfrak{g}}_\kappa$-modules on which c acts as the identity. Equivalently, defining*

$$U_\kappa(\hat{g}) \coloneqq U(\hat{g}) / (\mathbf{1} - 1)$$

this category is the category of modules over the completion $\overline{U}_\kappa(\hat{\mathfrak{g}})$ in the t-adic topology.

κ is called the *level* of the representation. One of the main goals of today's talk is to explain the special role of representations of *critical level.*

2 Finite Dimensional Case

Let G be a semi-simple simply connected Lie algebra, $\mathfrak{g} = \operatorname{Lie} G$, $B \subset G$ a Borel subgroup. G acts on $\mathfrak{g}\text{-mod}$ by $V \mapsto \operatorname{Ad}_g^* V$. We want to understand the categories

$$\operatorname{Rep} G = \mathfrak{g}\text{-mod}^G \subset \mathfrak{g}\text{-mod}^B \subset \mathfrak{g}\text{-mod}$$

The first category is the simplest: It is a semi-simple category with simple objects V_λ for each dominant weight λ. i.e.

$$\operatorname{Rep} G \simeq \bigoplus_{\lambda \in P^+} \operatorname{Vect}_\lambda$$

We want an analogous decomposition for the other two.

2.1 The center

For a DG-category $\mathcal{C}$, the center $Z(\mathcal{C})$ is by definition endomorphisms of the identity functor. It is a commutative algebra. For $x : T \to \operatorname{Spec} Z(\mathcal{C})$, let $I \subset Z(\mathcal{C})$ be the corresponding ideal. Define $\mathcal{C}_x$ the full subcategory where the action of $Z(\mathcal{C})$ factors through $Z(\mathcal{C})/I$.If $\mathcal{C} = A\text{-mod}$, we have an equivalence $Z(A\text{-mod}) \xrightarrow{\sim} Z(A)$ given by restriction.

Let $A = U(\mathfrak{g})$. Choose a basis $J_a; a = 1, \ldots, \dim \mathfrak{g}$ for $\mathfrak{g}$ and a dual basis J^a with respect to κ_0. The Casimir element

$$\Omega_1 = \frac{1}{2} \sum_a J^a J_a$$

lies in the center. The general form of the center is described via the Harish-Chandra homomorphism

$$Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W \hookrightarrow U(\mathfrak{h})$$

given by restriction of $U(\mathfrak{g}) \to U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{n} \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h})$.

$U(\mathfrak{h})^W$ is a polynomial algebra $\mathbb{C}[\Omega_i]$, $i = 1, \ldots, \operatorname{rank} \mathfrak{g}$, with $P_i \in U(\mathfrak{g})_{d_i+1}$, d_i the i-th exponent.

Thus $\operatorname{Spec} Z(\mathfrak{g}) = \operatorname{Spec} Z(\mathfrak{g}\text{-mod})$ is a finite dimensional affine space. Under certain integrability conditions, $\mathfrak{g}\text{-mod}$ will decompose into direct sum of the fibers over different points of this affine space. This is described in the next section:

2.2 Category $\mathcal{O}$

Let $\mathcal{O} = \mathfrak{g}\text{-mod}^{N,\mathrm{ss}} \subset \mathfrak{g}\text{-mod}^{N}$ be the subcategory where H acts semi-simply.

Let $\lambda \in \mathfrak{h}^*$. We get a map

$$\chi : Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W \to U(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}$$

χ is a well defined element of $\mathfrak{h}^*//W$. Denote $\varpi : \mathfrak{h}^* \to \mathfrak{h}^*//W$ the projection, so $\chi = \varpi(\lambda)$. Define $\mathcal{O}_\chi \subset \mathcal{O}$ the subcategory where the action of the center factors through the formal neighborhood of the closed point corresponding to χ.

We then have a decomposition

$$\mathcal{O} \simeq \bigoplus_{\chi \in \operatorname{Spec} Z(\mathfrak{g})} \mathcal{O}_\chi$$

.

The blocks generically have a simple description: For $\lambda - \rho$ dominant and anti-dominant, $\mathcal{O}_{\varpi(\lambda)}$ is a direct sum of $|W|$-categories, each equivalent to Vect and generated by the Verna module $M_{w\cdot\lambda}$. For $\lambda = -\rho$, it is a single copy of Vect generated by M_λ. For dominant integral λ, we can get nontrivial extensions between $M_{w\cdot\lambda}$ and $M_{w'\cdot\lambda}$.

If we add the requirement that $\mathfrak{h}$ acts with integral eigenvalues, we get the category $\mathfrak{g}\text{-mod}^B$. Thus the latter has a similar decomposition but with λ dominant and integral. Other blocks can be described as $\mathfrak{g}\,\mathrm{mod}^{\lambda,B}$ - B-integrable λ-twisted representations.

2.3 Flag Variety

We can use the geometry of the flag variety G/B to construct elements of $\mathcal{O}$: We have an action of G on G/B. Taking derivatives, we get a map $\mathfrak{g} \to \mathrm{Vect}(G/B)$. Let λ be a regular dominant weight. We have an extension

$$0 \to \mathcal{O}_{G/B} \to \mathcal{D}_{\leq 1,\lambda}(G/B) \to \mathrm{Vect}(G/B) \to 0$$

and thus a map

$$U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\varpi(\lambda)} \mathbb{C} \coloneqq U_{\varpi(\lambda)}(\mathfrak{g}) \to \mathcal{D}_\lambda(G/B)$$

from $\mathfrak{g}$ to twisted differential operators on G/B (defined as the quotient $\mathcal{D}(G)/\mathcal{D}(G) \cdot (\xi - \lambda(\xi); \xi \in \mathfrak{b})$).

Theorem 1. *(Beilinson-Bernstein Localization) The induces functor*

$$\mathrm{Loc} : U_{\varpi(\lambda)}(\mathfrak{g})\text{-mod} \to \mathcal{D}_\lambda\text{-mod}(G/B)$$

is an equivalence, with an inverse given by global sections Γ_λ.

We get an induced functor

$$\mathcal{D}_\lambda\text{-mod}(G/B)^B \to \mathcal{O}_{\varpi(\lambda)}$$

For a Schubert cell $Nw \subset G/B$, let $\delta(Nw) \coloneqq i_{Nw*}\mathcal{O}_{Nw}$. This is an N-invariant $\mathcal{D}_\lambda$-module, and so $\Gamma_\lambda(G/B, \delta(Nw)) \in \mathcal{O}_{\varpi(\lambda)}$. This is a *twisted Verma module*. This name is justified by the following:

Proposition 1. *For $w = w_0$, $\Gamma(\delta(wN)) = \Gamma(\mathrm{Fun}(N^-)) \simeq M_\lambda^\vee$. For $w_0 = 1$, $\Gamma(\delta(Nw)) = \Gamma(\delta_1) \simeq M_\lambda$*

If we further restrict to $\mathcal{D}_\lambda\text{-mod}(G/B)^G$, we get integrable modules: Indeed, if λ is integral dominant $\mathcal{D}_\lambda(G/B) \simeq \mathcal{D}(\mathcal{O}(\lambda))$. We have $\omega_{G/B} \in \mathcal{D}\text{-mod}(G/B)^G$ and by Borel-Weil-Bott:

$$\Gamma\left(G/B, \mathcal{O}(\lambda) \otimes \omega_{G/B}\right) \simeq \Gamma(G/B, \mathcal{O}(-\lambda)) \simeq V_\lambda$$

3 Vacuum and integrable representations

We want to generalize the above to the affine case. That is, a description of the center, that will give a decomposition of some affine category $\mathcal{O}$. We start by describing the flag variety. Unlike the finite case, there are several natural choices for a Borel subgroup. Several unequivalent choices are $G(\mathcal{O}), G(\mathbb{C}[t^{-1}]), N(\mathcal{K})T(\mathcal{O}), I$, where I is the preimage of B under the evaluation map $G(\mathcal{O}) \to G$. The first will correspond to Weyl / Verma modules, the second to the corresponding contragradient modules, and the lest to Wakimoto modules. We'll focus on the first: In that case the flag variety is $\mathrm{Gr}_G = G(\mathcal{O}) \subset G(\mathcal{K})$. This is a maximal parabolic subgroup of $\hat{G}$ regarded as an abstract Kac-Moody algebra. We have a global section functor

$$\Gamma^{\mathrm{IndCoh}} : \mathcal{D}_\kappa\text{-mod}(\mathrm{Gr}_G) \to \hat{\mathfrak{g}}_\kappa\text{-mod}$$

and thus a map

$$\mathcal{D}_\kappa\text{-mod}(\mathrm{Gr}_G)^{G(\mathcal{O})} \to \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$$

Taking $\delta_1 \in \mathcal{D}_\kappa\text{-mod}(\mathrm{Gr}_G)^{G(\mathcal{O})}$, we get

$$\mathbb{V}_\kappa := \Gamma_\kappa(\mathrm{Gr}, \delta_1) \in \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$$

This is the vacuum module of level κ. Explicitly:

$$\mathbb{V}_\kappa \simeq \mathrm{Ind}_{\mathcal{L}^+\mathfrak{g}\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}} \mathbb{C}$$

where $\mathcal{L}^+\mathfrak{g}$ acts by zero and $\mathbf{1}$ acts by the identity. If we replace $\mathbb{C}$ with any other V_λ, we get the Weyl module $\mathbb{V}_\kappa^\lambda$.

As for integrable modules: We have $\mathrm{Pic}(\mathrm{Gr}_G) \simeq \mathbb{Z}\mathcal{O}(1)$. If $\kappa = k \cdot \kappa_0$ is positive definite and integral, $\mathcal{D}_\kappa = \mathcal{D}(\mathcal{O}(k))$, and:

Theorem 2. *(Affine Borel-Weil-Bott)*

$$\mathbb{V}_{k\cdot\kappa_0}^{int} := \Gamma(\mathrm{Gr}_G, \mathcal{O}(k) \otimes \omega_{\mathrm{Gr}_G}) \simeq \Gamma(\mathrm{Gr}_G, \mathcal{O}(-k))$$

is an integrable module concentrated in degree 0, and is isomorphic to the unique maximal quotient of $\mathbb{V}_{k\cdot\kappa_0}$.

Remark. *In a similar way as above we can construct integrable modules $\mathbb{V}_{k\cdot\kappa_0}^{\lambda,int}$ for any dominant integral λ satisfying $\lambda\left(\hat{\theta}\right) \leq \ell$. We have a fully faithful embedding*

$$\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{K})} \hookrightarrow \mathfrak{g}\text{-mod}^G$$

sending $\mathbb{V}_{k\cdot\kappa_0}^{\lambda,int}$ to V_λ. In particular, this is a semi-simple category with finitely many simple objects.

4 The Center

4.1 The Casimir element

We're interested in the center

$$Z(\hat{\mathfrak{g}}_\kappa) \coloneqq Z(\hat{\mathfrak{g}}_\kappa\text{-mod}) \simeq Z\left(\overline{U}_\kappa(\hat{\mathfrak{g}})\right)$$

A first attemp will be to construct the loop of the Casimir element: For an element $X \in \mathfrak{g}$, let $X_n = X \otimes t^n$ and $X(z) = \sum_{n\in\mathbb{Z}} X_{(n)} z^{-n-1}$ (for now z^{-n-1} is just an index). Then we would like to define

$$\Omega_1^{\text{naive}}(z) = \frac{1}{2}\sum_a J^a(z) J_a(z)$$

The problem is that this is not even an element of $\overline{U}_\kappa(\hat{\mathfrak{g}})$: Take for example $\mathfrak{g} = \mathfrak{sl}_2$, so that $\Omega_1 = \frac{1}{2}\left(ef + fe + \frac{1}{2}h^2\right)$. Then

$$\Omega_1^{\text{naive}}(z) = \frac{1}{2}\left(e(z) f(z) + f(z) e(z) + \frac{1}{2} h(z)^2\right)$$

Take e.g. the first term:

$$e(z) f(z) = \sum_N \left(\sum_{n+m=N} e_n f_m\right) z^{-N-1}$$

but we only allow $m \to \infty$.

4.2 Fields and OPE

The solution for that comes from realizing $X(z)$ as an actual family of operators parametrized by points on a curve. Then interpret the product as a product of operators. We'll only allow operators satisfying a locallity conditions - those are called fields. The coefficients of z^n will then be elements of universal enveloping algebra, and in fact will generate a Lie-subalgebra.

Since we want our operators to vary along a curve, or at least a formal disk, we need them to be $\mathrm{Aut}(D)$-invariant. Thus we need an element of $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$. We'll thus take the vacuum $\mathbb{V}_\kappa$, i.e. $X(z) \in \mathrm{End}\,\mathbb{V}_\kappa((z))$.

Definition 3. *A collection of operators $\mathcal{F} = \{X(z) \in \mathrm{End}\,\mathbb{V}_\kappa((z))\}$ is a set of fields if they satisfy locallity: For $X, Y \in \mathcal{F}$, $X(z) Y(w)$ is well defined as an element of $\mathrm{End}\,\mathbb{V}_\kappa((z_1))((z_2))$. We then require that when taking the limit $z \to w$, we get a decomposition*

$$\lim_{z\to w} X(z) Y(w) =: X(w) Y(w) : + [X(z), Y(w)]$$

with $: X(z) Y(w) :$ regular in $z - w$ and $[X(z), Y(w)]$ is a sum of delta functions supported on the diagonal $z - w$. That is, it has a Taylor expansion with respect to $(z-w)^{-1}$.

Given two fields $X(z), Y(z)$, we define their product to be $: X(z) Y(z) :$ - this is the normally ordered product, and is again a field. The commutator $[X^1(z_1), X^2(z_2)]$ is the *chiral bracket* that will be discussed later.

Theorem 3. *(State-Field correspondence) For $X_{(-1)} \in t^{-1}\mathfrak{g} \subset \mathbb{V}_\kappa$, let $X(z) = X_{(-1)}(z) = \sum_n X_{(n)} z^{-n-1}$. By repeatedly applying normally ordered product, define $X(z)$ for any $X \in \mathbb{V}_\kappa$. Then the collection $\{X(z) : X \in \mathbb{V}_\kappa\}$ is a collection of fields on $\mathbb{V}_\kappa$*

We'll need the following consequence of locality:

Proposition 2. *For any $X, Y \in \mathbb{V}_\kappa$,*

$$[X(z), Y(w)] = \sum_{n \geq 0} \left(X_{(n)} \cdot Y\right)(w) \cdot (z-w)^{-n-1}$$

The above formula gives a Lie algebra structure on the Fourier coefficients, which is compatible with the Lie algebra structure on the universal enveloping algebra. In other words, if we denote by $\overline{L}(\mathbb{V}_\kappa)$ the completion of the span of all Fourier coefficients, that is the completion of symbols $A_{[m]}, A \in \mathbb{V}_\kappa, m \in \mathbb{Z}$ with respect to the topology defined by $m \to \infty$, we have a Lie algebra embedding

$$\overline{L}(\mathbb{V}_\kappa) \to \overline{U}_\kappa(\hat{\mathfrak{g}})$$

Definition 4. $\mathfrak{z}_\kappa(\hat{\mathfrak{g}}) \coloneqq \mathrm{End}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}_\kappa) \simeq (\mathbb{V}_\kappa)^{\mathcal{L}^+\mathfrak{g}}$

So we have a map $Z_\kappa(\hat{\mathfrak{g}}) \to \mathfrak{z}_\kappa(\hat{\mathfrak{g}})$. We'll start by constructing an element of $\mathfrak{z}_\kappa(\hat{\mathfrak{g}})$, and then show it lifts to $Z_\kappa(\hat{\mathfrak{g}})$.

4.3 Segal-Sugawara operator

We can now define the analog for the Casimir element: It will just be the field corresponding to the usual Casimir element (realized as an element of $\mathfrak{g} \otimes t^{-1}$)

$$\Omega_1(z) = \frac{1}{2} \sum_a : J^a(z) J_a(z) :$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, the first term will be

$$e(z) f(z) = \sum_N \left(\sum_{n+m=N, n<0} e_n f_m + \sum_{n+m=N, n \geq 0} f_m e_n \right) z^{-N-1}$$

Proposition 3. *For any J_b, we have*

$$\left[J^b(z), \Omega_1(z)\right] = \frac{\kappa - \kappa_c}{\kappa_0} J^b(z_2) \cdot (z_1 - z_2)^{-2}$$

In particular, all its Fourier coefficients $\Omega_{1,(m)}$ are in the center of $\mathrm{End}(\mathbb{V}_\kappa)$ precisely if $\kappa = \kappa_c$.

Proof. By 2, we have

$$\left[J^b(z), \Omega_1(z)\right] = \sum_{n \geq 0} \left(J^b_n \cdot \Omega_1\right)(z_1) \cdot (z_2 - z_1)^{-n-1}$$

Since Ω is of degree 2, any term with $n > 2$ will vanish. We're left with $n = 0, 1, 2$.

For $n = 2$:

$$\begin{aligned} J_2^b \cdot \frac{1}{2}\sum_a J_{-1}^a J_{a,-1} &= \frac{1}{2}\sum_a \left(J_{-1}^a J_2^b + [J_2^b, J_{-1}^a]\right) J_{a,-1} \\ &= \frac{1}{2}\sum_a J_{-1}^a \left[J_2^b, J_{a,-1}\right] + [J_2^b, J_{-1}^a] J_{a,-1} \\ &= \frac{1}{2}\sum_a J_{-1}^a \left[J^b, J_a\right]_1 + [J^b, J^a]_1 J_{a,-1} \\ &= \frac{1}{2}\sum_a \kappa\left([J^b, J^a], J_a\right) = \frac{1}{2}\sum_a \kappa\left(J^b, [J^a, J_a]\right) \end{aligned} \tag{4.1}$$

By choosing an orthogonal basis we can see that the last term equals zero.

For $n = 1$:

$$\begin{aligned} J_1^b \cdot \frac{1}{2}\sum_a J_{-1}^a J_{a,-1} &= \frac{1}{2}\sum_a \left(J_{-1}^a J_1^b + [J^b, J^a]_0 + \kappa\left(J^b, J^a\right)\right) J_{a,-1} \\ &= \frac{1}{2}\sum_a J_{-1}^a \kappa\left(J^b, J_a\right) + \left[[J^b, J^a], J_a\right]_{-1} + \kappa\left(J^b, J^a\right) J_{a,-1} \\ &= \sum_a \kappa\left(J^b, J_a\right) J_{-1}^a + \frac{1}{2}\sum_a \left[[J^b, J^a], J_a\right]_{-1} \end{aligned} \tag{4.2}$$

The first term is the decomposition of J^b by the basis J_{-1}^a, but multiplied by $\frac{\kappa}{\kappa_0}$. The second term is exactly the adjoint action of the Casimir element on J^b, thus equals $\frac{1}{2}\frac{\text{Kil}}{\kappa_0} = -\frac{\kappa_c}{\kappa_0}$. By a similar argument we can show the $n = 0$-th term vanishes, and so overall we have exactly the expression in the proposition.

□

Corollary 1. *$\Omega_{1,(m)} \in \mathfrak{z}_{\kappa_c}(\hat{\mathfrak{g}})$ for all m. Furthermore, since J_n^a generates the entire universal enveloping algebra, we get $\Omega_{1,(m)} \in Z_{\kappa_c}(\hat{\mathfrak{g}})$.*

For non-critical level, define $\tilde{\Omega}_m = \frac{\kappa_0}{\kappa - \kappa_c}\Omega_m$. By the above, we see that

$$[\tilde{\Omega}_1, J_n^a] = -nJ_n^a$$

That is - $\tilde{\Omega}_1$ is the grading operator. In particular we get:

Corollary 2. *For non-critical level the center $\mathfrak{z}_\kappa(\hat{\mathfrak{g}})$ is trivial.*

Proof. By induction on the length, one can show that an element is central iff it is killed by all $A_n, A \in \mathbb{V}_\kappa, n \geq 0$. In particular, $\tilde{\Omega}_1$ will act by zero. But this is the grading operator, so the central element must be the highest weight vector. □

In fact, one can show that the same is true for the entire center: For non-critical level $Z_\kappa(\hat{\mathfrak{g}})$ is trivial.

Corollary 3. *For* $\mathfrak{g} = \mathfrak{sl}_2$*,* $\mathfrak{z}_{\kappa_c}(\hat{\mathfrak{g}}) \simeq \mathbb{C}[\Omega_{1,(m)}]_{m \leq -2}$*, and* $Z_{\kappa_c}(\hat{\mathfrak{g}}) \simeq \mathbb{C}[\Omega_{1,(m)}]_{m \in \mathbb{Z}}$*.*

Proof. (sketch) For $m > -2$, $\Omega_{1,(m)}$ will act by zero on the highest weight vector, thus as an element of $\operatorname{End} \mathbb{V}_\kappa$ it is the zero operator. We've already constructed a map $\mathbb{C}[\Omega_{1,(m)}]_{m \leq -2} \to \mathfrak{z}_{\kappa_c}(\hat{\mathfrak{g}})$. To show it is an isomorphism, one shows it is an isomorphism on the associated graded. The latter is just the jet space $\operatorname{gr} \mathfrak{z}_{\kappa_c}(\hat{\mathfrak{g}}) \simeq JZ(\mathfrak{g}) \simeq J\mathbb{C}[\Omega_1] \simeq \mathbb{C}[\Omega_{1,(m)}]$. By the commutation relation any element $\Omega_{1,(m)}, m \in \mathbb{Z}$ will also commute with any J_n^a, and so we get a map $\mathbb{C}[\Omega_{1,(m)}] \to Z_{\kappa_c}(\hat{\mathfrak{g}})$, and again by a similar (but more involved) argument we can relate the associated graded of the latter with $LZ(\mathfrak{g})$, hence this map is an isomorphism. □

For a general $\mathfrak{g}$, we have $\Omega_n \in Z(\mathfrak{g})$ for $n = 1, \ldots, \operatorname{rk} \mathfrak{g}$. One can indeed show that a similar construction gives the entire center:

Theorem 4. $\mathfrak{z}_{\kappa_c}(\hat{\mathfrak{g}}) \simeq \mathbb{C}[\Omega_{n,(m)}]_{n=1,\ldots,\operatorname{rk}\mathfrak{g}, m \leq -2}$ *where* $\Omega_{n,(m)}$ *is the* m*-th Fourier coefficient of* $\Omega_n(z)$*. The center* $Z_{\kappa_c}(\hat{\mathfrak{g}})$ *is isomorphic to the completed topological algebra generated by all Fourier coefficients* $\Omega_{n,m}, m \in \mathbb{Z}$*.*

However, an explicit description of those elements is very difficult, and the proof is rather indirect. Frenkel-Feigin's original proof used a "free field realization" technique - embedding the vertex algebra $\mathbb{V}_{\kappa_c}$ into a vertex algebra associated to a commutative Lie algebra. Next week we'll see a shorted, geometric proof.